Quantum Theory: Ideals, Infinities and Pluralities

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Abstract

I argue that ideal, i.e. closed quantum systems, cannot exist because no system is completely isolated, they assume a radical object subject split, and their description involves an infinity which the world cannot instantiate. Inspired by the tension between transcendence and closure, I then argue that the Universe itself should not be described as a closed quantum system, because it would amount to an ultimate closure. If anything, the Universe should be conceived of as a plurality. I consider these questions in the light of mereological relations as well as epistemic features.

1 Overture

Emulating the maxim of enlightened despotism *Everything for the people but without the people*, the maxim of physics seems to be *Everything for reality but without reality.* The only case we can solve exactly is the ideal case, which tends to involve an infinity — the thermodynamic limit, zero temperature ($1/\infty$), particles coming from infinity, infinitely long plates… Then we master a conceptually delicate leap to the 'real' case, resulting in finite size effects, finite temperature, perturbations… I will argue that the ideal cannot exist in the physical world because it necessarily involves an infinity, and infinity cannot be supported in the world. The ideal, thus, acts as a ghostly pivot (as its existence is excluded from this world) from which we build a bridge to the real case. What kind of bridge is that?

The case that will be the centre of our attention is that of quantum theory, where closed quantum systems (the ideal case) can be used to provide a description of open quantum systems (the real case) via purifications. Quantum theory stands out among other theories in the fact that, in its appeal to the ideal, it need not involve an infinity in the dimension of the system. I will nonetheless argue that it needs to involve an infinity in its description, from where it will follow that closed quantum systems cannot exist in the world.

Then, inspired by the tension between transcendence and closure, I will argue that the Universe should not be considered as a closed quantum system. The most profound lesson of these considerations is, I believe, the contradictory nature of the notion of an ultimate closure, essentially because to be able to conceive of a unity implies putting a boundary, and that boundary could be transcended, frustrating the claim that the former unit was ultimate. Conceiving of the Universe as a closed system would amount to positing an ultimate closure. I will conclude that, if anything, the Universe should be conceived of as a plurality.

Before delving into this perspective, a disclaimer: I don't what is real, how to define reality, or if reality exists (I don't even know what it means to exist.) But I will pretend I do, in particular I will assume that there exists a physical reality. Throughout this text I will use 'real' as a synonym for 'physically relevant', which should mean that it can be instantiated in the physical world. 'Ideal' will be opposed to 'real'. The choice of words is unfortunate, as the real (in reference of reality) is in fact the ideal in the sense of the unattainable — what (mysteriously) draws and guides us. Another word of caution: I don't understand the physics of quantum theory, I just know its mathematics.

Let us start this journey by walking the bridge from the real to the ideal in quantum theory.

2 The Real and Ideal in Quantum Theory

As far as I see, ideal quantum systems cannot exist in the physical world at least because of three reasons. The first is ontological, the second is epistemic, and the third concerns the world's incapability to instantiate infinity.

The first reason is plain but far-reaching: No system can be completely isolated.

The second reason goes as follows. An ideal quantum system is assumed to be an object that can be defined and described in an isolated manner. So it does not only presuppose an ontic ideal (a perfectly isolated system) but also an epistemic ideal (whose state can be described perfectly). In particular, it posits a radical subject object split. If such a cartesian split is generally contested Gefter (2023), even more so in the case of quantum theory. At least during a measurement the quantum system must interact with the outside.[1]

In fact, the assumptions of complete isolation and the existence of another system that can perfectly describe the first one seem to contradict each other. For if a system were completely isolated, no other system (a.k.a. agent) could know anything about it. Knowing is, as far I see, a property of a physical system, and learning is a physical transformation. The many forms of knowledge and learning should have in common that a state of knowledge is expressible as a correlation between that very state (of the agent) and the state of the system, and learning expressible as the construction of such correlation. And correlating is only possible by means of an interaction between the system and the agent, which contradicts the purported total isolation of the system. Note that total isolation is understood as not only in the present but also in the past.

The third reason is that the mathematical representation of ideal quantum systems necessarily involves an infinity, and infinity cannot be supported in the physical world. Making this argument will occupy us for much of this section, concluding in 11. Through the journey, we will consider the relation of other physical theories with infinity, mathematics, as well as that of computer science. We will also shed a mereological and epistemological light on some of our digressions.

Note that, for the Universe itself, the first reason is moot, the second too (if the agent is to be outside the Universe), and the third unclear. I will argue in the next section, The

[1] If a subject is part of the very system it aims to describe, this results in forms of unpredictability, namely epistemic horizons of the subject (see, e.g., Fankhauser, Gonda, & De les Coves (2024)).

Universe as a Plurality, that the Universe should not be considered a closed quantum system for other reasons, grounded in the tension between transcendence and closure.

Let us set the stage.

The Real and Ideal in Quantum Theory In quantum theory, the ideal corresponds to closed quantum systems, which describe the situation where the physical system is completely isolated from the rest of the world, devoid of any interaction with the rest. Such systems are mathematically described by pure states. I stick to the interpretation by which a pure state, identified with a rank one density operator, represents perfect knowledge of the system. So far for the kinematics of quantum systems; with regard to the dynamics, the ideal corresponds to unitary evolution.

In quantum theory, the real corresponds to open quantum systems, which describe physical systems interacting with the rest. Mathematically, they are described by density operators. With regard to the dynamics, the real corresponds to quantum channels, that is, completely positive trace preserving linear maps.

As far as the mathematical formalism goes, the ideal is a special case of the real. In other words, the formalism of open systems is more expressive than for closed systems. Let me say a few a more things about the mathematical structure of our description of open quantum systems.

Open Quantum Systems, Mathematically Density operators are given by positive semidefinite matrices,[2] which embody a 'genuinely two dimensional notion' of positivity. Namely, nonnegativity is unambiguous for a number $x \geq 0$, which can be thought of as a zero dimensional array. For one dimensional arrays, the only meaningful notion of nonnegativity are nonnegative vectors, $v = (v_1, \ldots, v_n)$ such that $v_i \geq 0$ for all i. Two dimensional arrays can be nonnegative in the sense of (a) a nonnegative matrix, which is a matrix whose every entry is nonnegative, or (b) positive semidefinite, which is a genuinely matrix notion. Both notions form a cone, although with very different properties (see e.g. De les Coves, van der Eyden, & Netzer (2023)). I wonder if there is a three dimensional notion of nonnegativity, represented by cubes (a.k.a. rank-3 tensors), or higher dimensional notions. Why is the two dimensional notion of nonnegativity (provided by positive semidefinite matrices) so adequate to describe open quantum systems? As opposed to, say, three dimensional arrays. What is special about such two dimensional arrays? Note that the one dimensional notion is very useful to describe classical theories of nonnegativity, such as probability distributions, stochastic matrices, etc (see, e.g. Horn & Johnson (1985)).

Composition plays a central role in the theory of (open) quantum systems, coming to the foreground in the distinction between positive and completely positive maps, which is intimately linked with entanglement. In my eyes, what is surprising of complete positivity is that the condition seems to refer to an extrinsic property of the system where the map is acting, by involving an 'innocent bystander'. In fact, this is just the tip of the iceberg of the rich relations between the composition of systems and positivity. Mathematically, composition is given by the tensor product of vector spaces, resulting in a new vector space, whereas positivity forms a cone, and the tensor product of the cones is not the cone of the tensor products. The cone of positive semidefinite matrices is special in several ways; for example, it is self-dual De les Coves et al. (2023). If composition is important,

[2] I consider finite dimensions; similar considerations apply to infinite dimensions.

so is its converse, 'ignoring', mathematically modelled by the partial trace. As we will see below, this operation provides the bridge from the ideal to the real.

Turning to the dynamics of quantum systems, we encounter a contested issue (invisible in the kinematics): Whether transformations of open quantum systems should be required to be continuous in time. Surprisingly, most quantum channels do not describe a continuous time evolution; only a tiny minority do, i.e. they can be expressed as e^{tL} where L is Liouvillian of Lindblad form Wolf & Cirac (2008). An even tinier minority express *reversible* dynamics, namely those of the form e^{tL} where L only has a Hamiltonian part, rendering the evolution unitary. The only quantum channels whose inverse is also a quantum channel are precisely the ideal ones, i.e. unitary channels. Even more, quantum channels may not even be divisible once.[3] It follows that, *according to the theory of open quantum systems*, the evolution of a quantum system (whether closed or open) need not be reversible or continuous. I believe I understand why quantum channels describe irreversible dynamics: because they introduce some noise in the system that cannot be undone. Technically, they are non injective, because of the contraction following the purification. But why should they also be non continuous? The theory of open quantum systems, as far the dynamics is concerned, does not posit an infinite divisibility of time, as other physical theories do. In fact, the theory seems to inform us that time need not have a continuous nature.

Resorting to the Ideal in Quantum Theory Mixed states can be described without appealing to pure states, namely as positive semidefinite matrices, which are objects A such that $v^*Av \geq 0$ for all v. By the spectral theorem, they can also expressed as a convex combination of projections to pure states, $A = \sum_i p_i v_i v_i^*$, where this expression can be seen as a special case of Stinespring's dilation theorem. The appeal to the ideal followed by bridge to the real is transparent: The ideal takes the form of a rank one density operator followed by a contraction resulting in the convex combination.

Because of the isomorphism between linear maps and elements in a tensor product space, as well as some easy equivalences (known as the Choi–Jamiołkowski isomorphism), similar considerations apply to transformations of quantum states, where the unitary evolution corresponds to the ideal case and completely positive maps to the real case. Completely positive maps can be described directly as linear maps whose extension with the identity of any size[4] is positive, or via the Kraus decomposition, which is grounded on the same transparent appeal to the ideal via Stinespring's dilation. And, yet again, similar considerations apply to measurement operators described by positive operator valued measurements (POVMs; corresponding to the real case) versus projective measurements (corresponding to the ideal case). All these mathematical objects describing aspects of open quantum systems can be purified, i.e. given in terms of a modified ideal. This is extremely useful, if only conceptually surprising.

Such purifications imply that the description of open systems can be derived from that of closed systems, from which one may be tempted to conclude that closed systems are ontologically more fundamental than open systems, as the latter seem to be a 'special case' of the former (see Cuffaro & Hartmann (2021)). But this conclusion would be mistaken, I believe, for two reasons. One is that closed quantum systems do not exist in the physical

[3]For example, the qubit channel $\varepsilon(\rho) = (\mathrm{tr}(\rho)I + \rho^t)/3$ cannot be divided even once into two equal channels. This follows from the fact that its determinant is minimal Wolf & Cirac (2008).

[4]In fact, a size bounded by the dimension of the original space suffices. The significance of this fact will be discussed in the light of mererological relations (page 6.)

world, as I will try to argue below, rendering our fulcrum to the real world truly ghostly.[5] The second is that it embodies a reductionistic mindset, which has been proven not only narrow but also misleading on many fronts (we will consider emergence on page 10.) Taken to the extreme, this view may imply that culture and art are dispensable, as they strive for the singular whereas science strives for the universal, and the singular is in some sense derivable from the universal (De les Coves (2024)). Science is said to provide the view from nowhere, but I have a single dirty window from where to observe the world. While I somehow transcend this window with the help of reason, I believe there is value in the singular, precisely because it is unique and irreducible.

Let me end with two remarks. The first is that purifying has its own *mathematical* pitfalls. For example, purifying a mixed state may result in an uncontrolled increase of the cost of the representation, as measured by a rank De las Cuevas, Schuch, Perez-Garcia, & Cirac (2013).[6] A further difficulty of purifications, both conceptual and mathematical, is their non-uniqueness. Conceptually, it leads to interpretation ambiguities. By the very nature of the ideal case, we lack an operational criterion to pick one, as we can only access its 'shadow' (or projection or compression) to the real case. Mathematically, it renders the search for an optimal purification (in terms of cost) NP-hard.

The second remark concerns quantum magic squares, which are sets of POVMs that are compatible in a certain 'meshed' way. Their description cannot be obtained from the ideal: Not every quantum magic square can be purified. Technically, not every quantum magic square can be dilated to a quantum permutation matrix De las Cuevas, Drescher, & Netzer (2020).[7] What is the physical relevance of this result? Quantum magic squares should be related to strategies for quantum games (see e.g. Lupini, Mančinska, & Roberson (2020)), where the lack of a dilation has certain mathematical implications. However, the conceptual dimension of this fact still needs to be explored. The result suggests that quantum magic squares ought to be taken as fundamental, instead of quantum permutation matrices. In this case, there is a mathematical advantage of considering (the analogue of) the real case, as quantum magic squares are strictly more expressive than quantum permutation matrices together with their notion of contraction.

The Ideal in Physics Tends to Involve an Infinity In physics, the ideal tends to involve an infinity. Phase transitions are defined at the thermodynamic limit, $N \to \infty$ where N is the number of particles. Similarly, the zero temperature case plays a pivotal role, where 0 is seen as the infinitely small, $1/\infty$. (In this case there is even a principle of thermodynamics preventing this ideal from being realisable in the physical world, which does not prevent us from first solving the zero temperature case and then describing finite temperature as perturbations around the ideal case.) Ground states play an analogous role to zero temperature. We also often consider particles coming from infinity (as in the definition of the scattering matrix), infinite plates (say, with a certain charge density), perfectly round things, plane waves… All these idealisations involve an infinity.

If the ideal is represented by infinity and we contend that the real is represented by the finite, the relation from the real to the ideal becomes mereological: From a part to the whole. Writing $A \triangleleft B$ to denote that A is a part of B, we find the following relations analogous:

[5] Although such ghostly fulcrums are found elsewhere in physics, e.g. in thermodynamics.

[6] Although these separations seem to disappear in the approximate case De las Cuevas, Klingler, & Netzer (2021).

[7] The Science Breaker contains an invitation to this work.

In Physics: (♁)

Real $\lhd$ Ideal
Finite $\lhd$ Infinite

The latter infinity tends to be the actual in physics and potential in computer science, as we will discuss in page 8.

One remarkable thing about quantum theory is that the ideal need not involve an infinity, at least in the dimension of the quantum system. This follows from Stinespring's dilation theorem together with the Choi–Effros theorem.[8] It results in the following analogies:

In Quantum Theory: (♄)

Real $\lhd$ Ideal
Finite $\lhd$ Finite

The finite on the right hand side is of square dimension than to the left.

Why can we bound the dimension of the ideal quantum system? We can give a mathematical answer: because of the very stringent properties of completely positive linear maps. But is there a deeper, conceptual reason? I don't know. One could search within attempts to single out quantum theory of other generalized probabilistic theories. (There may be similar bounds in other areas of physics that I am unaware of — in this case, I would be grateful if the reader shared them with me.) Be it as it may, mathematics seems to inform us of the properties of the ideal, and why should we let that happen?

I want to make the case that the description of the ideal in quantum theory necessarily involves an infinity. First note that the dimension of the Hilbert space is observable only after an unbounded number of measurements of identical copies. This involves two infinities: one is the number of copies, and the second is the identical condition, which involves infinite precision. So the fact that the dimension of the Hilbert space of the ideal quantum system is finite does not imply that it be accessible with a finite number of physical operations. Second, an ideal quantum system represents in a limiting case, that of complete isolation, that of no interaction whatsoever with the outside. For this reason, its description must involve arbitrary precision: the interaction between the system and the environment must be smaller than *any* given precision. If this mathematical description is to be consistent with that of open quantum systems and every other area of physics I can think of, the statement of 'any given precision' can only be represented by involving an infinity. In particular, it involves the potential infinity (by division) and, in our usual mathematical formulation, the actual infinity (see page 8). In contrast, open quantum systems are not associated to such radical views, neither ontic (the system could have a dependence, even if tiny, on the environment) nor epistemic (they allow me to represent my uncertainty). As a consequence, their mathematical descriptions need not involve an infinity. For example, a density operator could be given in terms of rationals with a finite precision (see page 8).

Let us now consider infinity's relation with the world (with our finite and fallible minds).

Infinity and the World Can the physical world support infinity? I want to answer 'no': The world cannot instantiate infinity. To that end, I will partially rely on Aristotle's

[8]Recently generalized in De les Coves et al. (2023).

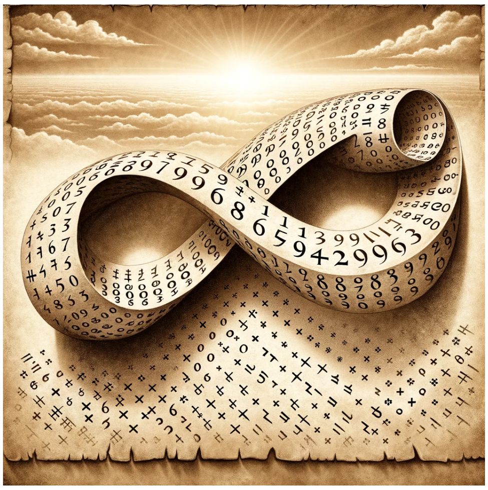

Figure 1: ChatGPT's rendering of the potential infinity by addition of Aristotle. I believe it only begins to capture the beauty of some of his ideas, and the scope of infinity itself. But who or what can capture infinity if all there is are finite things?

answer (Fig. 1). He distinguished between potential and actual infinity, and argued that the potential infinite can exist in the physical world but the actual infinity cannot (Moore, 2019, Cha. 2). Namely, he defended that there is no objection to something's being infinite in the world provided that its infinity is not there all at once.[9]

In modern terms, I take the potential infinity to be the unbounded, and the actual infinity to be (isomorphic to) an infinite set, that is, a set whose cardinality is at least $\aleph_0$. This aligns with Priest's argument that every potential infinity needs the actual infinity to be defined by the so-called Domain Principle (Priest, 2001, Cha. 8). The potential infinity aligns with Hegel's 'false infinity', and the actual infinity to his 'true infinity' (Priest, 2001, Cha. 10).

As far as I understand, Aristotle's reasons for accepting the potential infinity's physical existence were grounded in his belief that time was infinite by addition — he assumed that time is unbounded in the past and the future. With today's knowledge, the first seems to be false and the second is unclear. The space of the Universe also seems to be finite. Hence I conclude that the potential infinity cannot be supported in time or space. And, if the foregoing conclusion is true, I posit that potential infinity cannot be supported at all in the physical world. With regard to potential infinity by division, either in time or space, I do not have any reason to think they can be instantiated in the physical world. Why should I believe that something can be divided arbitrarily many times? This something could be empty space, or time. Infinity is a construct with such crazy properties (page 16) that, in the absence of a reason to believe they can be exemplified in the world, I find it is more sober to believe they cannot. This will be crucial to argue why ideal quantum systems cannot exist (page 11).

Despite Aristotle's conclusion,[10] Greek geometry presupposes infinite space by addition and by division — for example, a line is indefinitely extensible and infinitely divisible. On the other hand, arithmetic presupposes the existence of infinitely many natural numbers,

[9]The potential infinity can be imagined like a clock endlessly ticking. Its ticking is potentially but never actually infinite. The ticking is in a constant state of becoming, but never actually *is*, in its entirety. It never achieves full being Moore (2019).

[10]Please see (Moore, 2019, Cha. 1) for the beautiful and moving ideas on infinity by Anaximander, Pythagoras, Parmenides and Plato.

since each natural number has an immediate successor but no natural number is itself infinite. This rationale was then exported to axiomatization, whose main appeal is its pretension to trap an infinite wealth of information or wisdom in a finite, manageable stock of basic (self-evident) principles. These hopes were capped by Gödel (see page 13).

The foregoing considerations seem to beg the question of the relation between mathematics, more generally, and the world. In contrast to infinity, some numbers — say, the number three — seem to instantiated in the world, despite the fact that the number three as such does not exist in the physical world. (That 'three' is instantiated in a collection of (three) pencils, fingers or moons means somehow that we identify 'three' with the equivalence class of all things which are three.) Similarly, relations such as $3 + 2 = 5$ seem to be instantiated in the physical world, as we never observe that three oranges together with two oranges make anything other than five oranges. Geometrical objects such as circles, lines or points are only poorly instantiated — probably because their precise instantiation would require supporting infinity.

In my eyes, the relation between mathematics and the world is a particular case of a broader problem, the problem of universals in metaphysics (see e.g. Loux & Crisp (2017)). Using the term (and communicating the meaning) 'three' is not so different from uttering other lexical items (such as 'run', 'scissors', 'sleep' or 'kitchen'). It requires recognizing unity in infinite diversity. This seems to be very close to the core of the problem of cognition. It is also related to the emergence of meaning, to which we shall briefly return in page 14 (for broader considerations, see e.g. Eco (1997)).

The Theory of Physics and Infinity Despite the world's inability to sustain the actual or potential infinity (as argued above), physical theories heavily rely on infinity — usually, the actual infinity. The mathematical framework employed in physical theories is profoundly grounded in the concept of infinity. For example, we describe space, time, energy or temperature with real numbers. I believe that the potential infinity, both by addition and division, would in fact suffice, as we only need to invoke finitely many multiples or fractions of fundamental units, such as the meter, the kilogram, or the second. Such fractions or multiples should be finite but unbounded, leading to rational multiples of the fundamental units. And each fundamental unit need not involve an infinity, as it is taken as a 'brute fact' stemming from the world which we use as a yardstick (pardon the redundancy) to measure other such worldly facts.

Let me make the same point from a different angle. Most will agree that what is physically accessible (i.e. what is currently measurable) are rational multiples of the fundamental units with a certain precision. Since we presume that precision will improve with time (as we happen to live in a magnificent epoch of constant discovery), it would be absurd and soon obsolete to develop a theory for a specific precision. Granting that, we would want a physical theory that accounts for *any* precision, that is, that aims to describe rationals multiples of the units, instead of real (in the sense of $\mathbb{R}$) multiples, which are their completion. Each real is the limit of a (in fact, many) sequences of rationals, e.g. for example, π, e, $\sqrt{2}$ or literally countless others. It is uncontested that the notion of a limit is *mathematically* very powerful — in analysis, continuity, derivatives and integrals rely on limits, and more generally, any infinitely dimensional object (whether it be a group, an algebra, a field, a vector space, or any other mathematical object.) But we should not be dazzled by the bright light of infinities in mathematics and mistake them for something real.

Del Santo & Gisin (2019) have recently challenged the use of real numbers in physics.

They find that if classical physics were only to use quantities admitting a finite description, it would be indeterministic. So they regard real numbers as the hidden variables of classical mechanics, in analogy with the search for hidden variables in quantum mechanics.

In quantum theory, we use 'twice' the reals (in an intertwined way), namely the complex. Renou, Trillo, Weilenmann, Thinh, Tavakoli, Gisin, Acin, & Navascues (2021) have recently questioned the need of complex numbers, or better said, of imaginary numbers in quantum physics.[11] But why not use *hyperreals*, instead of reals, also with an imaginary part? The hyperreals are an extension of the reals Goldblatt (1998) which allow one to solve long-standing questions in quantum information, such as the existence of non-trivial tensor stable positive maps, which are intimately related to the existence of NPT bound entanglement Van der Eyden, Netzer, & De las Cuevas (2022). The only superficial virtue I see in the reals over the hyperreals is that the former are simpler, so may seem preferable by Occam's razor. But the two have the same type of infinity ($\aleph_1$), and I would argue that reals are already 'full of monsters' — for example, essentially no element in $\mathbb{R}$ admits a finite description, and how can we even conceive of something infinite whose only description is itself? More reflections on reaching infinity from the finite can be found in De les Coves, Corominas-Murtra, & Solé (2024).

Physics' artificial relation with infinity comes to the foreground when it aims to connect with computer science, which relies on the potential infinity — for example, the computational complexity of a physical problem can only be studied if the problem is discretized.

The Potential Infinity in Computer Science The theory of computation is based on the distinction between the finite and the potential infinite. A Turing machine consists of a finite number of transformation rules, where each symbol is an element of a finite alphabet. The crucial attribute to these elements *finite*: Finite things are combined in a mechanistic way. The only component that is unbounded is the length of the tape. That a Turing machine halts means that it *eventually* halts, that is, after a finite but unbounded time. 'Dually', a formal language is defined as $L \subseteq \Sigma^*$ where Σ is a finite alphabet and Σ^* the set of concatenations of elements of Σ *of any length.* Σ^* represents the unbounded. If L is finite, its complexity is trivial within the standard theory of computation, because there is a Turing machine whose transition rules are precisely the set of elements of L, so the machine accepts L in one time step. In fact, a finite state automaton without loops suffices. I propose referring to such machines as *dull*, because they distort the very idea that an algorithm is a construct that enables to reach infinity from the finite. Indeed, I see the theory of computation as a systematic study of which infinite sets can be described with a finite number of rules. Note that this concerns the potential infinite by addition, not by division.

Yet how meaningful is the notion of a halting machine *physically*? Not very, I contend. The Turing machine is required to *eventually* hold, there is no bound at 590875 billion — just finite. Physically this is ridiculous, as the requirement of finite (without any further qualification) corresponds to no time scale, which is very unphysical. Some will counter that this is the rationale behind the class P, which asks the machine to halt in polynomial time. I believe they are partially right, but I would add P relies on two requirements

[11] Why should the fantastic structure of complex numbers help us describe the quantum world? 'Fantastic' because it was conceived as something imaginary which required the introduction of a symbol i to solve certain algebraic equations. Why the minuscule world should be so beautifully described by this mathematical construct is a mystery to me.

which involve an infinity. The first is asymptotic scaling, by which the length of the string going to infinity. The second is worst case analysis, by which the hardest of infinitely many instances of the problem determines its complexity. I believe the reason for requiring these conditions is, again, mathematical instead of physical. For example, average case complexity has an uglier mathematical structure — e.g. the 'harder-than' relation need not be transitive Wigderson (2019).

It follows from these considerations, together with the physical world's incapability to sustain infinity, that it is perilous, if not flawed, to measure the complexity of physical problems based on the theory of computational complexity. One such pitfall are the undecidability results, e.g. Wolf, Cubitt, & Perez-Garcia (2011); Cubitt, Perez-Garcia, & Wolf (2015). More generally, I am unclear on the importance of computational complexity results for physics (and I have personally worked on them for quite some time, e.g. Klingler, van der Eyden, Stengele, Reinhart, & De las Cuevas (2023)). Ultimately, we are asking: What relation should the theory of physics have with the potential infinity? In page 8 I argued that it may be sensible for physical theories to aim to describe the potential infinity (not only by addition but also by division).

Both physics and computer science strive for the infinite from the finite, from a certain perspective, in a similar way: Local physical interactions can be seen as the *grammar* of physics Stengele, Drexel, & De las Cuevas (2023); Reinhart & De las Cuevas (2022). This pursuit is not exclusive to these domains of enquiry. In the cognitive sphere, Quine argued that it was only because there are infinitely many things that we need to operate with the fundamental notion of a thing at all (Moore, 2019, Cha. 9). For this notion is used principally in making generalizations, by specifying, one by one, what each was like. This, again, lies at the heart of the problem of universals in metaphysics.

Epistemologically What can be better known, the real or the ideal? Mathematically, the ideal over the real — I think this is the very *raison d'être* of the ideal. Together with (♁) this seems to imply that:

In Physical Theories (Appealing to the Ideal):

Knowledge of the part $<$ Knowledge of the whole

But this was too fast. For, in reductionistic approaches, the parts can be better known than the whole, probably because they do not appeal to ideals:

In Reductionistic Physical Theories:

Knowledge of the part $>$ Knowledge of the whole

Much of condensed matter physics, approximation techniques, etc are concerned with bridging this gap. A more or less good knowledge of the parts degrades as they are composed, as the description becomes too difficult (there are too many parameters or degrees of freedom) and we lose control over the whole.

As a matter of fact, condensed matter physics is riddled with emergent phenomena, associated with the appearance of new degrees of freedom which behave differently than the underlying ones.[12] It follows that the knowledge of the whole we are after is essentially

[12] The inadequacy of reductionism is one of the driving forces behind the research field called Complex Systems. See e.g. Solé & Goodwin (2000); Solé (2009); Holland (2014). One of the most striking examples of the failure of reductionism is the brain itself: we understand well how pairs of neurons behave, but how does cognition, consciousness emerge?

independent of the knowledge of the parts. In my eyes, emergence is an example of the fact that a theory may be more ontologically fundamental than another, but have have *less* explicatory power — in terms of identifying the relevant degrees of freedom, explaining their behavior, and predicting.

In quantum theory, the observation that the ideal can be better known than the real, together with ($\hbar$), implies that

In Quantum Theory:

Knowledge of the part < Knowledge of the whole

The (lack of) knowledge on either side is optimal for maximally entangled states, which describe perfect knowledge of the whole and perfect ignorance of the parts (in fact, this is one way to define these states.) Even more, by the purification theorem (page 4), any lack of knowledge of the state of the system can be attributed to it being entangled with an unspecified, so far extrinsic environment, so that together they form an 'epistemic perfect' whole.

Closed quantum systems involve an infinity, which cannot be supported If the description of closed quantum systems necessarily involves an infinity (page 6), and infinity cannot be supported in the physical world (page 7), it follows that closed quantum systems cannot exist in the physical world. This is the third reason why they cannot exist (cf. page 2). As far as I see, the argument mainly concerns the relation of this world with certain abstractions, namely those involving infinities.

Now, some have argued that infinity could be detectable in the physical world by means of the recent result of Ji, Natarajan, Vidick, Wright, & Yuen (2020). They provide a non-local game whose winning value is 1 if the strategy of the two players is of a certain kind, and at most 1/2 if it is of another kind. The two players are in possession of a quantum system each, such that the strategy of the first kind can be implemented only if their quantum systems are of infinite dimension. Hence, if it were possible to measure the value of this game, and the result were such that one could conclude it is 1, it would follow that their respective dimension was infinite. This would amount to witnessing infinity in the physical world! There are many challenges (some perhaps unsurmountable) to realise and measure such a physical system. But, if it were possible to do so, my third reason would be false. And, among others, the Church–Turing thesis would become obsolete.

Let us now turn to our house, the Universe.

3 The Universe as a Plurality

Can you imagine reaching the end of the Universe and stretching an arm outwards (Fig. 2)? And traveling back in time to the very first moment of the Universe, and then steeping further back to the moment before? Can you imagine conceiving the inconceivable? And thinking the unthinkable?

All these contradictions have a common skeleton: They try to reach beyond a limit from within that very place. On one hand, they are within a place — be it physical or non-physical, like space, time, the set of caused events or of thinkable things. On the other hand, they aim to reach beyond this place, to exit it, to transcend it — to stretch an arm beyond the edge, to exist before the beginning of time, to apprehend the inconceivable, to think the unthinkable. The first condition is called *closure* and the

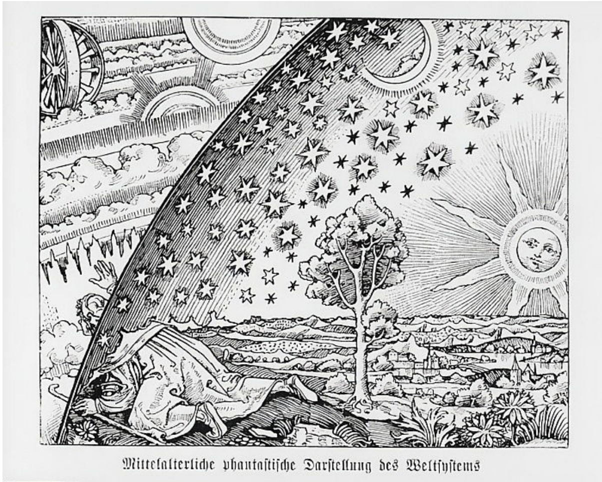

Figure 2: If the Universe had an edge, success in stretching out an arm would show that there was at least empty space beyond; failure that there was something preventing it. Either way, this would not, after all, be an edge. This is the contradiction issued by claiming an ultimate closure.

second *transcendence* Priest (2001). To transcend a condition while being subject to that condition is a contradiction.

Many philosophical questions can be seen through the lens of the tension between transcendence and closure Priest (2001). Many thinkers fall in the trap of affirming that something is unreachable, inconceivable, unthinkable, ineffable — and the trap becomes a whirlpool that pulls the logical ground from beneath their feet. For instance, for Parmenides, what is not cannot be nor be thought of — yet here we are, thinking about it at this very moment. If skepticism contends that 'Nothing can be known', then the assertion made by this statement should itself be unknowable. If relativism posits that 'Everything is relative', then this assertion should also be subject to relativity. A statement like 'Something is indescribable' is problematic, because it ascribes a quality ('indescribable') to that entity, thereby contradicting the claim that it cannot be described at all. For example, Kant asserts that the noumenon cannot be predicated, but we just predicated it — asserting that there are entities beyond our judgment is, in itself, a form of a judgment about them. Similarly, Aristotle's prime matter is all in potentiality and nothing in particular, it is pure possibility, contains all properties while itself having no property — but this is contradictory because we are currently assigning a property to it. Or Quine argues and communicates that meaning is inscrutable, yet this statement should make this very communication impossible. The very adjective 'ineffable' embodies a contradiction.

This taste of contradictions illustrates the many manifestations of transcendence and closure. Priest identifies them at the limits of cognition, conception, expression, and iteration. My aim was not to quote these thinkers with precision, but rather to demonstrate the many ways how one might attempt to reach the end of some Universe and stretch out the arm — this absurdity proposed by Archytas has many reincarnations.

I will rely on the failure to conceive of an ultimate closure in order to argue that the Universe should be conceived, if anything, as a plurality, and in particular, not as a closed quantum system. Let us start by examining the skeleton of the tension.

Transcendence, Closure and Existence The tension is structured as follows:

Something is contained within a set (Closure)
This something transcends this set (Transcendence)
This something exists (Existence)

As a set, we can imagine the collection of things with a certain property. This skeleton is called the *Inclosure Schema* or *Schema T* Priest (2001); Bolander (2017).

Transcendence, Closure, and Existence imply a contradiction. In mathematics or any study built on logic, contradictions tend not to be admitted. This is mainly due to the principle of explosion in consistent logics, by which a contradiction allows to derive any statement as true: If A and not A, then everything follows. Therefore, if we find a contradiction, we reject it. To do so, we conclude that at least one of the three (C, T, or E) is false — usually Existence, as the others tend to be true by construction. Some, notably Priest, appeal to accepting the contradiction and adopting a so-called paraconsistent logic, where the principle of explosion does not hold. But the prevailing approach is to deny the existence of the object that makes the contradiction possible. We will see examples in matters of infinity below.

Yet, the mandates of logic may not be as rigid in philosophy, or may be open to scrutiny. How we traverse the muddy terrain without disappearing in the mire will depend on the specific question, but at the core we will have the problem that to be conceived and to be bounded are essentially the same thing Priest (2001). Because if something is bounded then it can be conceived as everything that is on this side of the boundary, and if something is conceived then it is bounded by the terms of the conception. In a very naive example, if the Earth were flat, envisioning the magnificent edge that bounds the oceans would involve, even if just implicitly, contemplating what may lie beyond this boundary. I think that art, especially, confronts contradiction by forsaking the claim to universality De les Coves (2024).

Let us now see how to stretch out an arm at the end of the Universe.

I am a liar A systematic way of escaping closure and transcending is through self-reference and negation. If I say

I am a liar

you will conclude that I am lying if and only if I am not lying. Similarly, if we consider the statement

This sentence is false

we will conclude that it is true if and only if it is false. If we analyze this sentence with the same tools used to formulate it, we are not be able to give it a consistent value of truth or falsehood. In this sense, this sentence transcends the set of true or false sentences.

Similarly, Gödel demonstrated the first incompleteness theorem by constructing a sentence in an axiomatic system that says

I am not provable. (☽)

We again conclude that we cannot prove it or disprove it — it transcends provability. By Gödel's theorem, one of the most profound results in mathematics, formal systems with a minimum level of expressiveness cannot be consistent and complete at the same time.

'Consistent' means that we cannot prove a false sentence as true, and 'complete' that we can prove all true sentences. Some have concluded that truth is a notion greater than provability; others that our brain is more powerful than a machine because we can grasp the paradox (but this conclusion may be premature Fraser, Solé, & De las Cuevas (2021)).

More formally, (☽) reads 'the sentence whose number corresponds to this very sentence is not provable'. A crucial ingredient is the assignation of 'handles' or 'names' (called Gödel numbers) to sentences, which then serve as variables, so that sentences can *talk about* sentences. Gödel numbering allows for reference, similarly to how the code of the Turing machine can be written in a tape and fed to another Turing machine, or how sets are given a 'name' or 'tag' in Cantor's diagonalization argument.[13] If reference is the first step, the next one is self-reference: If reference is expressive enough, it can include itself. Self-reference composed with a function without a fixed point, such as negation, leads to contradictions which tend to result in limitations.

The versatility and scope of self-reference and negation are remarkable, partly because the paradox cannot be 'fixed' (see e.g. Bolander (2017)). I summarize it with the maxim

No system can talk about itself. (☼)

Here, 'talk' means 'describe all attributes'—systems can partially talk about themselves, that is, describe some attributes (in fact, this is a common compromise when confronting these limitations). 'Talk' also means 'talk consistently,' so that one cannot simultaneously affirm an attribute and its opposite.

Let me explain (☼) in the simplest case, that of a set, where it coincides with Cantor's theorem. For a system, we take a set S. If we identify an attribute with its extension, i.e. the set of elements where it is true, then an attribute is an element of the set of subsets of S, written 2^S. For example, if S is the set of natural numbers and f is the attribute 'even' so that $f(n) = 1$ if n is even and $f(n) = 0$ if not, we identify f with the set of even numbers,

$$\{2, 4, 6, 8, 10, 12, \dots\}.$$

Cantor's theorem states there is no way to pair the elements of S with those of 2^S without leaving an element of 2^S unmatched. S cannot encompass 2^S, its attributes, that is, S cannot talk about itself. *No* S can encompass 2^S—no matter how wild and intrincate S is, 2^S will be more so. It follows that there are infinite types of infinities! If one infinity blows our mind as if touching the stars, imagining infinite types of infinity is like flying to the end of the Universe and coming back to explain it.

So Cantor's theorem transforms the ancient liar paradox into a technique called diagonalization, which, for sets, allows tearing any membrane of purported closure and transcending it. It implies that there is no such thing as the set of all sets, there is no largest infinity, there is no ordinal that contains all ordinals, there is no ultimate closure—one can always transcend Priest (2001); Moore (2019). For other mathematical objects, Lawvere's fixed point theorem Lawvere (1969); Yanofsky (2003); Gonda et al. (2024) is, as far as I know, the most precise and powerful formulation of the scope of transcendence and closure, as it expresses the conditions that a system must satisfy to be prey to these contradictions. This theorem specializes in Cantor's theorem, Gödel's theorem, the uncomputability of the halting problem, Russell's paradox, and more, which express

[13] This pattern can be made explicit by means of the simulator of Gonda, Reinhart, Stengele, & De les Coves (2024). The emergence of reference amounts to a transfer of information from the target T to the context C; this is radical for Turing machines, where a single object T (the universal Turing machine) can run any algorithm.

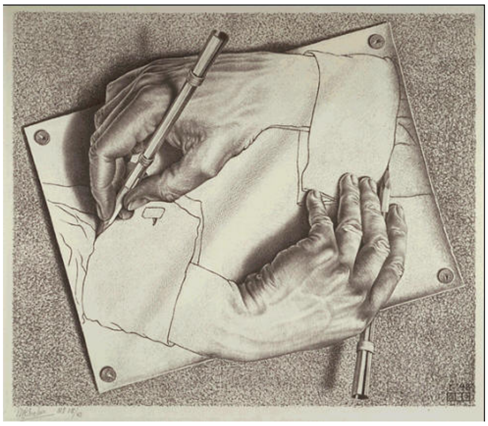

Figure 3: Which hand draws which? A tangled hierarchy from the (undrawn) hand of Escher.

deep and irreparable limitations in provability, computability, the notion of set, learning, etc. All have at their core a contradiction of self-reference and negation that forces them to conclude that certain objects cannot exist. The consequences are devastating across mathematics, computer science, and any enterprise nourished by logic.

Now forget everything I just said and consider (☼) again. It seems to me that I can think about myself without encountering insurmountable walls, and don't you think you can do the same? Either something about (☼) is suspicious, or the human condition relates to this limitation in a very surprising way. I have argued elsewhere (De les Coves (2024)) that we carry the contradiction between transcendence and closure in the syntax of our being, that being human consists of struggling to enclose infinity in a finite body, that we transcend and enclose ourselves — that the person, the *prosopon*, the one behind the mask, is both part and whole at the same time.

Let us now tear down the hierarchies and entangle them.

Tangled Hierarchies Graphically, self-reference and negation can be understood as a tangled hierarchy, which is a hierarchy that 'closes in on itself,' meaning that the highest level ends up being influenced by the lowest level. In fact, the demonstration of Lawvere's theorem — which is very short — consists of a diagram that captures the tangled hierarchy generating the contradiction. Hofstadter (1979) identifies in *Gödel Escher Bach* this tangled hierarchy in Gödel's proof, works of Escher (Fig. 3 or Fig. 4), and some works of Bach, where the 'ornaments' of the piece of music evolve into the primary structure. In formal systems, the axioms and transformation rules are expected to determine the truth or falsehoold of any well-formed sentence in that system. Yet, Gödel's sentence (☼) says something meaningful about the very axioms that give rise to it (in particular, it expresses a limitation). Something remarkable has occurred: Reference has emerged within the axiomatic system. Gödel's is not a sterile sentence whose truth can be proven or disproven from the axioms, but it *points* somewhere, in particular it refers to itself.[14]

Surprisingly, some aspects of the functioning of DNA form a tangled hierarchy, too Hofstadter (1979). Because DNA is expressed in proteins and determines their composition, it seems to be at a higher hierarchical level. Yet some proteins determine which segments of the DNA are expressed — so they seem to be at a higher level. We find a tangled

[14]Hofstadter calls it the emergence of meaning, but I find 'reference' more accurate.

Figure 4: Is the person at the art gallery or at the coastal village? Another tangled hierarchy from the hand and mind of Escher.

hierarchy in the reading and translation of the alphabet that materially conforms us, in the smallest units (of meaning) in our very packaging.

In formal sciences, entangling a hierarchy is ‘bad’ because formal systems form a hierarchy where everything is derived from the (unjustified) truth of axioms. (Imagine a world resting on a giant turtle that is supported by nothing — this is the role of axioms). If we entangle the hierarchy and find a contradiction, we conclude that some assumption is false, for example, the existence of an object. But in other areas, entangling a hierarchy can be good or magical, because it may reveal a mechanism of emergence, as if rising from the ground by pulling one’s own shoelaces. In fact, the tangled hierarchy is Hofstadter’s proposal to explain the emergence of the self and consciousness, as explained more clearly in *I Am a Strange Loop* Hofstadter (2007).

The tension between transcendence and closure should be expressible as a tangled hierarchy, where transcendence corresponds to the higher level and closure to a lower one. This connection would underline the idea that we must embrace contradictions, since, if Hofstadter is right, they are very close to the self — so close that they logically constitute us.

Infinity: the Part Embraces the Whole What defines an object? What parts make up a whole? These very significant questions (in my opinion) occupy metaphysics (see e.g. Carroll & Markosian (2010)).

Consider a whole and a proper part of this whole, that is, a part that is neither empty nor the whole itself — for example, a part of a leg, a part of a fish, or a part of a rainbow.[15] By definition, a proper part is a subset of the whole, in the sense that there are elements of the whole that do not belong to the proper part. It seems evident that the elements of the proper part cannot be paired with those of the whole, because the proper part is *smaller* than the whole, and thus the part cannot encompass the whole.

In the infinite, a proper part can encompass the whole. Formally, each element of the part can be paired with an element of the whole. For example, in the natural numbers,

[15] While this may appear obvious to us, it may not have been so in the past or for other thinkers. For example, Leibniz postulated a very awkward mereology of the world where everything is composed (in a nuanced way) of monads, and each monad mirrors the whole Universe (see e.g. Antognazza (2016)).

we can pair each even number $2n$ with a natural number n and leave no natural number unpaired, because *there is no last element*. This is very bizarre: a part of a mountain is different from the whole mountain, and the same goes for a cow, a nail, or any physical thing. In the infinite, a proper part can be equivalent to the whole (in a formal sense). The whole transcends the part in the sense that there are elements of the whole that are not in the part (for example, the odd numbers), but at the same time, the part achieves closure because it encompasses the whole (with the pairing). In fact, the existence of a surjection from a proper part to the whole is a hallmark of infinity. Because infinity has 'no boundary', such outlandish things can happen. Priest calls it a paradox at the limit of the iterable.

The Universe as a Plurality The most profound lesson of transcendence and closure is that there is no ultimate closure, or, if there is one, we cannot conceive of it because we would have to imagine a membrane around it, which would brush against the very thing we are denying. Lawvere's theorem provides a systematic way to open a door of the purported final house and step out into the street. The collection of houses (closures) and streets (their transcendence) cannot be seen as a new house; there is no such thing as the set of all sets. *There are multiplicities that cannot be conceived as a unit* — they cannot be understood as a whole and therefore cannot be considered 'a thing'. Cantor called these multiplicities inconsistent totalities; I call them streets and houses that cannot be grouped in a new house.

Some say that the only closed system that truly exists is the Universe as a whole. I feel very uncomfortable making any such claims. If anything, I want to push against the hopes (or aspirations) that reality should be conceived of as a unity and not a plurality. If a set is a many which allows itself to be thought of as one, and some collections are too large to be considered as one, how could the Universe be considered as one? We should not hope for a last refuge, an ultimately closed system. On the epistemological side (and almost ethical side too), this stance is grounded in my finiteness, which instils in me a profound sense of humility. It makes me sympathetic to the view that, if the Universe can be modelled as anything (an extremely ambitious conditional), it should be according to the open systems view. But the question whether the Universe should be described as a closed or open system is probably moot, as both views presuppose that quantum mechanics holds at the scale of the Universe — an unlikely fact, in my eyes.

I disagree with Einstein when he says Einstein (2023):

> We must escape the chaos of contradictory fragmentary events and useless passions that produce nothing but restlessness. We can only free ourselves from the chaos that surrounds us by creating with reason scientific or ethical rational systems (...) Only with reason can we free ourselves from transience and approach the eternal through objectivity.

And I agree with Pessoa (2020) (or his heteronym Alberto Caeiro):[16]

> One excessively clear day,
> a day that made one wish to have worked hard
> so as not to have to work at all that day,

[16] Twenty years ago I read several books of Pessoa's heteronyms and selected only this poem, I don't know why. Recently, I encountered it again and it acquired an ironically consistent meaning within my understanding of transcendence and closure.

I caught a glimpse, like a path between the trees,
of what might very well be the Great Secret,
the Great Mystery that the false poets talk about.

I saw that there was no Nature,
that Nature does not exist,
that there are valleys, mountains, plains,
that there are trees, flowers, grasses,
that there are rivers and stones,
but that there is no whole to which all this belongs,
that the real and true ensemble
is a mania of our ideas.

Nature is made of parts without a whole.
Perhaps this is the mystery they speak of.

And it was this that, without thinking or pondering,
I realized must be the truth
that everyone tries to find and does not find,
and that only I, by not having sought it, have found.

The Universe is too wild to be considered one. We ought to approach such magnificent pluralities with awe, humility and fantasy.

4 Coda

What would physics without the mathematical ideal look like? Del Santo & Gisin (2019) offer some hints. What would physics centered around the real look like? The theory of open quantum systems provides some clues. What would mathematics without infinity be like? It may resemble constructivism. Interestingly, the pivot on the ideal seems to be alien to biology — limits to infinity or asymptotic reasoning do not seem to be of help.

Abandoning the reference to the ideal is, in my eyes, analogous to Popper's shift in epistemology, where the old question 'What is truth?' is replaced by 'How can sources of error be recognised and corrected?'. Instead of referring to the ideal, he places himself 'in the middle of the sequence' and enquiries how to improve his knowledge. This gives rise to falsifiability, which, ironically, is our methodological paradigm. Similarly, 'What is justice?' is replaced by 'How can we improve the political system without violence?' This gives rise to democracy, presumably. Eschewing the appeal to the ideal, in matters of justice, is a move whose importance cannot be overstated. History has told us that justice ideals (utopias) can be highly non-unique and very different for different people.

What is a mystery is to me is how we — confused, perishable, open systems — can conceive of the ideal, for example, of infinity. But this mystery may again be just be a different guise of the power of the mind to abstract and to unify, exercised whenever reference takes place.

What is clear is that there is no knowledge without a conceptual framework. That is why initiatives like this book are so important, because we should not take anything for granted, let alone what is supposed to be fundamental.

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